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Manifolds and Group actions

Homework 7

Mandatory Exercise 1. (5 Points)
Let G be a Lie group acting smoothly on a manifold M.

a) Show that if G is compact that the action is proper.

b) Show that if G is not compact and M is compact that the action is not proper.

Mandatory Exercise 2. (10 Points)
Let $\mathcal{H} = \{X \in \mathrm{Mat}_n(\mathbb{C}) \,|\, X = X^*\}$ be the set of Hermitian matrices. Recall that Hermitian matrices have real eigenvalues and are diagonalizable by unitary matrices. For $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ denote by $\mathcal{H}_\lambda$ the set of Hermitian matrices with spectrum λ.

a) Check that the formula

$$A \cdot \xi := A\,\xi\,A^{-1}, \quad A \in U(n), \quad \xi \in \mathcal{H}$$

defines an action of $U(n)$ on $\mathcal{H}$

b) Show that the orbits of this action are the spaces $\mathcal{H}_\lambda$.

c) Compute the stabilizer of a diagonal matrix with entries $\lambda_1 = \lambda_2 = \ldots = \lambda_k < \lambda_{k+1} = \lambda_{k+2} = \ldots = \lambda_n$.

d) Show that the stabilizer of $\xi \in \mathcal{H}$ is conjugate to the stabilizer of $A \cdot \xi$.

e) Discuss the conjugacy class of a stabilizer of an element of $\mathcal{H}_\lambda$ for any λ.

f) Let λ be such that $\lambda_1 < \lambda_2 = \ldots = \lambda_n$. Do you recognize $\mathcal{H}_\lambda$ as a familiar space?

Mandatory Exercise 3. (5 Points)
Consider $SO(3)$ and its lie algebra $\mathfrak{so}(3)$. Prove that

$$\mathfrak{so}(3) = \left\{ \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \,\middle|\, (a_1, a_2, a_3) \in \mathbb{R}^3 \right\}$$

a) Show that $\mathfrak{so}(3)$ is isomorphic to $(\mathbb{R}^3, \times)$ as a Lie algebra, where $\times$ denotes the exterior product on $\mathbb{R}^3$.

b) Show that $\mathfrak{so}(3)$ is isomorphic to $\mathfrak{su}(2)$ as a Lie algebra.

Hand in preferably on 8th of June via email, or 12th of june in the pigeonhole on the third floor.